ON A CONJECTURE OF SEBASTIÁN MARTÍN RUIZ

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1. The conjecture

Sebastián Martín Ruiz proposed his conjecture 44 on `www.primepuzzle.net` in this form

Let p, q be consecutive prime numbers, $p < q$. Let

$$z = \sqrt{\frac{p^2 + q^2}{2} - 1} \tag{1}$$

Conjecture: p, q are prime twins iff z is integer.

2. Reformulation

Following equation is equivalent to (1)

$$2(z^2 + 1) = p^2 + q^2 \tag{2}$$

We forget for the moment that p and q are prime. The left side is even and thus the right side need also to be even. In other words p and q need to have an even difference. Being $p < q$ we can set $q = p + 2s$. The value z is also obviously larger than p and we can set $z = p + r$. Equation (2) can be rewritten as

$$\begin{aligned} 2\left((p+r)^2 + 1\right) &= p^2 + (p + 2s)^2 \\ &= 2p^2 + 4ps + 4s^2 \\ &= 2(p^2 + 2ps + 2s^2) \end{aligned}$$

becoming

$$p^2 + 2pr + r^2 + 1 = p^2 + 2ps + 2s^2$$

or

$$2p(r - s) = 2s^2 - r^2 - 1 \tag{3}$$

We are now interested in all solutions of (3) with p, r, s positive integers. When $r = s$ then $s^2 - 1 = 1$, i.e., $s = 1$ and for each p (not only prime), $q = p + 2$ gives in (1) an integer $z = p + 1$. Thus obviously, if p and q are twin prime, then z is integer.

A simple search shows that for all small primes p (say < 1000000) if p and q are consecutive prime and $q > p + 2$, then z is not integer.

Suppose thus that $r > s$, and set $r - s = x$. It follows $r = s + x$. Equation (3) now becomes

$$\begin{aligned} 2px &= 2s^2 - s^2 - 2sx - x^2 - 1 \\ &= s^2 - 2sx - x^2 - 1 \end{aligned} \tag{4}$$

or

$$s^2 - 2sx - x^2 - 1 - 2px = 0 \tag{5}$$

Solving by s we get

$$s = x \pm \sqrt{2x^2 + 2px + 1} \tag{6}$$

In (6) the value under square root is larger that x^2, and as s need to be positive, it follows that only the sign + can be considered.

Thus $s \gg \sqrt{p}$.

It is conjectured that the gap between primes is $\ll p^{0.525}$ (see for example Guy, Unsolved Problems in Number Theory, A8), but this is not enough. Anyway, a counterexample needs a gap not smaller than $2\sqrt{2p+3}+2$, and even if such a gap can be realized, it is still not say that then z is an integer.

In the same reference it is also said that the Riemann Hypothesis implies that the gap is $< p^{1/2+\epsilon}$, and this would possibly imply the conjecture of Sebastián Martín Ruiz, although some details need to be precised.

Again in the same reference, Dorin Andrica conjecture is cited: for consecutive primes $p < q$

$$\sqrt{q} - \sqrt{p} < 1.$$

From this it would follow that

$$q - p < \sqrt{q} + \sqrt{p}.$$

Latter, with Bertrand Postulate, i.e., $q < 2p$, we would get

$$q - p < 2\sqrt{2p}$$

implying the conjecture of Sebastián Martín Ruiz.

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