Proof of Conjecture 75

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1 Introduction

In 2009 Sebastián Martin Ruíz empirically got a formula for the Carmichael Function $\lambda(m)$, when m is the factorial $n!$.

$$\lambda(n!) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2, \\ 2 & \text{if } n = 3, \\ 4 & \text{if } n = 5, \\ \frac{n!}{2 * n\#} & \text{otherwise.} \end{cases} \tag{1}$$

where $n\#$ is the greatest primorial $p_k\#$, such that $p_k \leq n$. Here we prove that the conjecture is true.

2 Preliminaries

We know the following holds for any m:

$$\lambda(m) = LCM\Big[\lambda(p_k^{\alpha_k})\Big]_k, \tag{2}$$

where m is decomposed into its prime factors: $m = \prod_k p_k^{\alpha_k}$. We also have for any prime $p > 2$:

$$\lambda(p^\alpha) = (p-1)p^{\alpha-1}. \tag{3}$$

Finally for powers of 2 and $\alpha > 2$ we have:

$$\lambda(2^\alpha) = 2^{\alpha-2}. \tag{4}$$

3 Proof

We can easily check that the conjecture holds for $n \leq 5$. We now show that it holds for $n > 5$. Let $n! = \prod_k p_k^{\alpha_k}$ then from Equation 2:

$$LHS = \lambda(n!) = LCM\Big[\lambda(p_k^{\alpha_k})\Big]_k = LCM\Big[\lambda(2^{\alpha_1}), \ldots, \lambda(p_k^{\alpha_k})\Big]. \tag{5}$$

We now use Equation 3 and since $n > 5$ we can also apply Equation 4:

$$LHS = LCM\Big[2^{\alpha_1 - 2}, \ldots, (p_k - 1)p_k^{\alpha_k - 1}\Big]. \tag{6}$$

In the above, terms $(p_k - 1)$ will never be a prime above p_k, so they will not contribute to the LCM and hence can be ignored. So we have an LCM of powers of primes, which simply becomes the product of these powers of primes:

$$LHS = 2^{\alpha_1 - 2} \prod_{k \geq 2} p_k^{\alpha_k - 1} = \frac{\prod_k p_k^{\alpha_k - 1}}{2} = \frac{\prod_k p_k^{\alpha_k}}{2 \prod_k p_k}. \tag{7}$$

The above is simply $\frac{n!}{2 * n\#}$, which is what we needed to prove.