Minimal symmetrical tuples of consecutive Twin primes

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April 4, 2020

It is the purpose of this article to give some insight into the number of minimal symmetrical tuples of Twin primes with length $2n$ less than a given bound x. We will use the Hardy-Littlewood conjecture to estimate the number of primes that comply to the associated admissible prime constellation. A final correction is necessary to make sure that these primes are actually consecutive.

1 Definition

A symmetrical tuple of consecutive twin primes is a list of $2n$ consecutive primes $[p_1, p_2, \cdots, p_{2n-1}, p_{2n}], n \in \mathbb{N}$ such that:

- The list constists of pairs of twin primes: $p_{2k} = p_{2k-1} + 1, k \in \{1, \cdots, n\}$
- The list is symmetrical (around its center): $p_k + p_{2n+1-k} = c$ for some constant c
- The primes in the list are consecutive.

The tuple is called minimal if its diameter $p_{2n} - p_1$ is minimal. Such a tuple is called a prime constellation.

2 Admissible constellations

If, for the moment, we drop the requirement that the primes are consecutive, we may first focus on finding conditions on primes $p \in \mathbb{P}$ and a list of numbers a_k where $p_k = p + a_k$ is prime:

$A_n = [a_1, a_2, \cdots, a_{2n}]$

We are to find a constellation, thus multiple values of p, such that all the p_k are prime. This means that we can't find a divisor q (other than the p_k). The residue of p_k modulo q should not equal 0. This is equivalent to finding a_k such that the set:

$A_{n,q} = \{a_1 \mod q, \cdots, a_{2n} \mod q\}$

contains at least one free residue modulo q. For every prime q there are q residues to consider. This means that if $q > 2n$ there is at least one free residue. In order to find a condition on p it is therefore sufficient to consider $q \leq 2n$. For every q_i we obtain a set of free residues S_{q_i}. An application of the Chinese remainder theorem may give us a set of residues modulo the product of the primes q_i, from which we may choose p.

It is not my intention to find these constellations. These are precomputed by Natalia Makarova[1][2].

2.1 $A_2 = [0, 2, 6, 8]$

The reduced residue sets are:

q	$A_{2,q}$
2	$\{0\}$
3	$\{0, 2\}$

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[1] Natalia Makarova, A_3 privately communicated

[2] Problem 73, primepuzzles.net

This means that we should have:

$$\begin{cases} p \equiv 1 \mod 2 \\ p \equiv 2 \mod 3 \end{cases}$$

And the Chinese remainder theorem (or just trying) gives: $p \equiv 5 \mod 6$.

2.2 $A_3 = [0, 2, 12, 14, 24, 26]$

The reduced residue sets are:

q	$A_{3,q}$
2	$\{0\}$
3	$\{0, 2\}$
5	$\{0, 1, 2, 4\}$

This means[3] that we should have:

$$\begin{cases} p \equiv 1 \mod 2 \\ p \equiv 2 \mod 3 \\ p \equiv 2 \mod 5 \end{cases}$$

And the Chinese remainder theorem gives: $p \equiv 17 \mod 30$.

The given constellation does not contain $[4, 6, 8, 10, 16, 18, 20, 22]$. Given the previously computed condition $p \mod 30 = 17$, we have $p + \{4, 10, 16, 22\} \mod 3 = \{0\}$; or 3 is a divisor in these cases. We further have $p + \{8, 18\} \mod 5 = \{0\}$, so 5 is a divisor in these cases. We additionally need $p + \{6, 20\}$ to be not prime in order for the constellation to consist of consecutive primes.

2.3 $A_4 = [0, 2, 12, 14, 18, 20, 30, 32]$

q	$A_{4,q}$
2	$\{0\}$
3	$\{0, 2\}$
5	$\{0, 2, 3, 4\}$
7	$\{0, 2, 4, 5, 6\}$

This means that we should have:

$$\begin{cases} p \equiv 1 \mod 2 \\ p \equiv 2 \mod 3 \\ p \equiv 4 \mod 5 \\ p \equiv \{4, 6\} \mod 7 \end{cases}$$

And the Chinese remainder theorem gives: $p \equiv \{179, 209\} \mod 210$.

The given constellation does not contain $[4, 6, 8, 10, 16, 22, 24, 26, 28]$. Given the previously computed residues, we have $p + \{4, 10, 16, 22, 28\} \mod 3 = \{0\}$, $p + \{6, 26\} \mod 5 = \{0\}$. Only $[8, 24]$ might generate an additional prime, which we would like to avoid since we seek consecutive primes. If $p \equiv 4 \mod 7$, $p \equiv 179 \mod 210$ we should additionally have that $p + 8$ is not prime and similarly if $p \equiv 6 \mod 7$, $p \equiv 209 \mod 210$ we should impose that $p + 24$ is not prime.

2.4 $A_5 = [0, 2, 6, 8, 18, 20, 30, 32, 36, 38]$

We should have:

$$\begin{cases} p \equiv 1 \mod 2 \\ p \equiv 2 \mod 3 \\ p \equiv 1 \mod 5 \\ p \equiv 2 \mod 7 \end{cases}$$

And the Chinese remainder theorem gives: $p \equiv 191 \mod 210$.

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[3] Find the missing residues in the complement of the displayed residue sets. For instance $\{0, 1, 2, 4\} \mapsto \{0, 4, 3, 1\}$ thus residue $2 \mod 5$ is missing

2.5 $A_6 = [0, 2, 12, 14, 24, 26, 30, 32, 42, 44, 54, 56]$

Define the primorial function $p_k\# = \prod_{m=1}^{k} p_m$.
The residues of $p \mod p_5\#$, $p_5\# = 2310$, are in the set:

$\{587, 797, 1007, 1247, 1457, 1667\}$.

2.6 $A_7 = [0, 2, 24, 26, 30, 32, 42, 44, 54, 56, 60, 62, 84, 86]$

The residues of $p \mod p_6\#$ are in the set:

$\{797, 3737, 10037, 14657, 15287, 19907, 26207, 29147\}$.

2.7 $A_8 = [0, 2, 30, 32, 42, 44, 54, 56, 60, 62, 72, 74, 84, 86, 114, 116]$

The residues of $p \mod p_6\#$ are in the set:

$\{6047, 14627, 15287, 23867\}$.

2.8 $A_9 = [0, 2, 18, 20, 30, 32, 42, 44, 60, 62, 78, 80, 90, 92, 102, 104, 120, 122]$

The residues of $p \mod p_7\#$ are in the set:

$\{150089, 240179, 270209, 360299\}$.

2.9 $A_{10a} = [0, 2, 12, 14, 30, 32, 42, 44, 54, 56, 90, 92, 102, 104, 114, 116, 132, 134, 144, 146]$

There are 108 residues of $p \mod p_8\#$:

$\{59987, 96947, 203207, 217067, 360287, 397247, 517367, 623627, 660587, 780707, 817667, 923927, 937787, 1081007, 1238087, 1314317, 1538387, 1614617, 1771697, 1801727, 1928777, 1958807, 2035037, 2192117, 2335337, 2455457, 2492417, 2635637, 2649497, 2755757, 2792717, 2912837, 2949797, 3056057, 3069917, 3176177, 3213137, 3333257, 3370217, 3476477, 3513437, 3633557, 3670517, 3776777, 3790637, 3933857, 4090937, 4167167, 4197197, 4354277, 4391237, 4511357, 4654577, 4811657, 4887887, 5044967, 5188187, 5308307, 5345267, 5502347, 5532377, 5608607, 5765687, 5908907, 5922767, 6029027, 6065987, 6186107, 6223067, 6329327, 6366287, 6486407, 6523367, 6629627, 6643487, 6749747, 6786707, 6906827, 6943787, 7050047, 7063907, 7207127, 7244087, 7364207, 7507427, 7664507, 7740737, 7770767, 7897817, 7927847, 8084927, 8161157, 8385227, 8461457, 8618537, 8761757, 8775617, 8881877, 8918837, 9038957, 9075917, 9182177, 9302297, 9339257, 9482477, 9496337, 9602597, 9639557\}$.

2.10 $A_{10b} = [0, 2, 12, 14, 42, 44, 54, 56, 60, 62, 84, 86, 90, 92, 102, 104, 132, 134, 144, 146]$

There are 16 residues of $p \mod p_8\#$:

$\{143357, 263477, 877517, 1284497, 1360727, 2381747, 2501867, 3522887, 6176657, 7197677, 7317797, 8338817, 8415047, 8822027, 9436067, 9556187\}$.

2.11 $A_{11} = [0, 2, 12, 14, 30, 32, 42, 44, 54, 56, 72, 74, 90, 92, 102, 104, 114, 116, 132, 134, 144, 146]$

There are 16 residues of $p \mod p_8\#$:

$\{360287, 660587, 780707, 1801727, 2192117, 3213137, 3333257, 4354277, 5345267, 6366287, 6486407, 7507427, 7897817, 8918837, 9038957, 9339257\}$.

2.12 $A_{12} = [0, 2, 30, 32, 42, 44, 60, 62, 72, 74, 84, 86, 120, 122, 132, 134, 144, 146, 162, 164, 174, 176, 204, 206]$

There are 40 residues of $p \mod p_9\#$:

{6906797, 12492377, 16486367, 22192067, 26186057, 26306177, 36005867, 42732587, 45585437, 52432277, 64984817, 69669497, 71831657, 74684507, 79369187, 91231037, 95225027, 98768567, 100930727, 104924717, 118167947, 122161937, 124324097, 127867637, 131861627, 143723477, 148408157, 151261007, 153423167, 158107847, 170660387, 177507227, 180360077, 187086797, 196786487, 196906607, 200900597, 206606297, 210600287, 216185867}.

2.13 A_{13}

There are 4 different admissible constellations, giving a total of 197 residues of p modulo $p_9\#$.

3 Hardy-Littlewood conjecture

The length of the admissible constellation $2n$ defines some prime q_{max}, the largest prime q for which $q \leq 2n$. The admissible initial values of p are confined to a set of residues modulo a primorial $q_{max}\#$, thereby greatly reducing the amount of work in finding a solution for an admissible constellation, i.e. finding an explicit values for p. On the downside the number of conditions on p have increased, it should have effect on the number of possible solutions p under a given bound x. The Hardy-Littlewood conjecture adresses this point. As the name suggests it is not a theorem, it is based on reasonable assumptions. It can be derived using a probabilistic argument, as shown in Prime Numbers and Computer Methods for Factorization (Riesel,1994). I will use a more direct approach, for which I'll use the article 'Hardy-Littlewood constants' by Keith Conrad [4]. See also [5].

Given $f_i(t) = t + a_i$, with a_i a term defined by the constellation, define:

$$\pi_{f_1,\cdots,f_{2n}}(x) = \#\{k \leq x : f_1(k), \cdots f_{2n}(k) \text{ are all prime}\}$$

This function counts the number of prime constellations, with the smallest prime less than or equal to x. We need the various a_i to be distinct and $f(k)$, the product of the $f_i(k)$, may not be divisible by a prime q for all integers k, i.e. we need a free residu a_i modulo q for all values of i, q.

Conjecture) (Hardy-Littlewood):

$$\pi_{f_1,\cdots,f_{2n}}(x) \overset{c}{\sim} C(f_1,\cdots,f_{2n})\frac{x}{\ln(x)^{2n}} \text{ or } \pi_{f_1,\cdots,f_{2n}}(x) \overset{c}{\sim} C(f_1,\cdots,f_{2n})\int_2^x \frac{1}{\ln(t)^{2n}}\mathrm{d}t$$ [6]

where:

$$C_f \doteq C(f_1,\cdots,f_{2n}) = \prod_{q\in\mathbb{P}} \frac{1 - w_f(q)/q}{(1-1/q)^{2n}}$$

and $w_f(q)$ is the number of roots of $f(t) = f_1(t)\cdots f_{2n}(t)$ in $\mathbb{Z}/q\mathbb{Z}$.

In other words: to compute the constant C_f we have to find the number of occupied residues modulo a prime q within our constellation defined by a_i, for every prime q.

In order to compute this constant we need to split the primes q. If $q > a_{2n}$ we know that the values of $f_i(t) \mod q = f_i(t)$ are distinct and we have $w_f(q) = 2n$. This leads to a socalled $2n$-tuple constant of Hardy-Littlewood, which covers the 'large portion' of the primes q:

$$C_{2n} = \prod_{q\in\mathbb{P},q>a_{2n}} \frac{1-2n/q}{(1-1/q)^{2n}} = \prod_{q\in\mathbb{P},q>a_{2n}} \frac{q^{2n-1}(q-2n)}{(q-1)^{2n}} = \prod_{q\in\mathbb{P},q>a_{2n}} \frac{\dfrac{q-2n}{q}}{\left(\dfrac{q-1}{q}\right)^{2n}}$$

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[4] Hardy-Littlewood constants, Keith Conrad

[5] On The Asymptotic Density Of Prime k-tuples and a Conjecture of Hardy and Littlewood , Lásló Tóth

[6] We have $\int_2^x \frac{1}{\ln(t)^k}\mathrm{d}t \sim \frac{x}{\ln(x)^k}\left(1 + \frac{k}{\ln(x)}\right)$, for large x

The second product has $q-2n$ in the numerator, the number of free residues. The last product is a consequence of a probabilistic argument, it displays a fraction of two probabilities. In the numerator the probability that the $t+a_i$ are prime and that these events are dependent, divided by the probability that each of the $t+a_i$ are prime and that these events are independent.

For $q \leq a_{2n}$ we have to compute the number of free residues for each prime q separately. (I'll use F, the contribution of a finite number of q):

$$F_{a_{2n}} = \prod_{q\in\mathbb{P}, q\leq a_{2n}} \frac{q^{2n-1}\cdot \#\text{free residues} \mod q}{(q-1)^{2n}}$$

For the given constellations we already computed the number of free residues for $q \leq q_{max}$. We also need to check the (possible) values $2n < q \leq a_{2n}$

Thereby splitting the requested constant into two parts: $C_f = F_{a_{2n}} \cdot C_{2n}$

Example - The twin prime constant

The admissible constellation is $p+[0,2]$. We find that $q_{max}=2$. The twin prime constant, or 2-tuple constant equals

$$C_2 = \prod_{q\in\mathbb{P}, q>2} \frac{1-2/q}{(1-1/q)^2} = \prod_{q\in\mathbb{P}, q>2} \frac{q(q-2)}{(q-1)^2} = \prod_{q\in\mathbb{P}, q>2} \left(1-\frac{1}{(1-q)^2}\right) \approx 0.6601618158^7$$

We necessarily have $p \mod 2 = 1$. There is one free residue and find:

$$F_2 = \frac{2^1\cdot 1}{(2-1)^2} = 2.$$

4 Computation of Hardy-Littlewood constants C_n

The question is how to effectively compute an infinite product over the primes:

$$C_n = \prod_{q\in P, q\geq q_0} \frac{q^{n-1}(q-n)}{(q-1)^n}$$

The index n is changed for convenience, we'll use even n later on. We necessarily have $q_0 > n$. Lets change the product into a sum by applying the natural logarithm:

$$\ln(C_n) = \ln\left(\prod_{q\in\mathbb{P}, q\geq q_0} \frac{q^{n-1}(q-n)}{(q-1)^n}\right)$$

$$\ln(C_n) = \sum_{q\in\mathbb{P}, q\geq q_0} \ln\left(\frac{q^{n-1}(q-n)}{(q-1)^n}\right)$$

$$\ln(C_n) = \sum_{q\in\mathbb{P}, q\geq q_0} \ln\left(\frac{q^n(1-n/q)}{q^n(1-1/q)^n}\right)$$

$$\ln(C_n) = \sum_{q\in\mathbb{P}, q\geq q_0} \left(\ln(1-n/q) - n\ln(1-1/q)\right) \tag{4.1}$$

We now apply the series representation of $\ln(1-x) = -(x + x^2/2 + x^3/3 + \cdots)$:

$$\ln(C_n) = \sum_{q\in\mathbb{P}, q\geq q_0} \left(-(n/q + n^2/(2q^2) + n^3/(3q^3) + \cdots) + n(1/q + 1/(2q^2) + 1/(3q^3) + \cdots)\right)$$

$$\ln(C_n) = \sum_{q\in\mathbb{P}, q\geq q_0} \sum_{j=2}^{\infty} \frac{n-n^j}{j}\frac{1}{q^j} = -\sum_{j=2}^{\infty} \frac{n^j - n}{j} \sum_{q\in\mathbb{P}, q\geq q_0} \frac{1}{q^j} \tag{4.2}$$

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[7] The on-line encyclopedia of integer sequences, A005597

We may write:

$$\sum_{q\in\mathbb{P},q\geq q_0}\frac{1}{q^j}=P(j)-\sum_{q\in\mathbb{P},q<q0}\frac{1}{q^j} \tag{4.3}$$

Where $P(s)$ equals the prime zeta function:

$$P(s)=\sum_{q\in\mathbb{P}}\frac{1}{q^s} \tag{4.4}$$

In a similar manner, by using the Euler product definition of the Riemann zeta function, one may derive:

$$\ln(\zeta(s))=-\sum_{q\in\mathbb{P}}\ln(1-q^{-s})=\sum_{k=1}^{\infty}\frac{P(ks)}{k}$$

Inversion using the Moebius function $\mu(\cdot)$ gives:

$$P(s)=\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\ln(\zeta(ks)) \tag{4.5}$$

If we substitute (4.5) into (4.3) and finally in (4.2) we arrive at:

$$\ln(C_n)=-\sum_{j=2}^{\infty}\frac{n^j-n}{j}\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\ln(\zeta(kj))-\sum_{q\in\mathbb{P},q<q_0}\frac{1}{q^j}\right) \tag{4.6}$$

Thus converting the task of computing an infinite product to summing an infinite numbers of values of the Riemann zeta function. Since time is limited, we need to bound both series to say j_{max}, k_{max}. The good news is that good algorithms exist in order to compute $\zeta(\cdot)$.

The convergence may be enhanced[8] by splitting (4.4)

$$P(s)=\sum_{q\in\mathbb{P},q<q_1}\frac{1}{q^s}+\sum_{q\in\mathbb{P},q\geq q_1}\frac{1}{q^s}$$

And subsequently alter $\zeta(s)$ to:

$$\zeta_{q\geq q_1}(s)=\zeta(s)\prod_{q\in\mathbb{P},q<q_1}\left(1-\frac{1}{q^s}\right) \tag{4.7}$$

Which leads to $\ln(\zeta_{q\geq q_1}(s))=O(q_1^{-s})$, this gives a handle to influence the speed of convergence, by suitably choosing q_1 large enough. And (4.4) is generalized to:

$$P(s)=\sum_{q\in\mathbb{P},q<q_1}\frac{1}{q^s}+\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\ln(\zeta_{q\geq q_1}(ks)) \tag{4.8}$$

If we assume that $q_1>q_0$ we may write:

$$\ln(C_n)=-\sum_{j=2}^{\infty}\frac{n^j-n}{j}\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\ln(\zeta_{q\geq q_1}(kj))+\sum_{q\in\mathbb{P},q_0\leq q<q_1}\frac{1}{q^j}\right) \tag{4.9}$$

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[8] High precision computation of Hardy-Littlewood constants, Henri Cohen

If we take a close look at (4.9) we may observe that after splitting the right hand side, or by retracing the steps all the way back to (4.1) that we may write:

$$\ln(C_n) = \sum_{q\in\mathbb{P},q_0\le q<q_1} (\ln(1-n/q) - n\ln(1-1/q)) - \sum_{j=2}^{\infty} \frac{n^j - n}{j} \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(\zeta_{q\ge q_1}(kj)) \right) \tag{4.10}$$

The sum on the left hand side is computed for a limited number of primes q. One may even choose $q_1 = q_0$ with the understanding that this sum is set to 0. The sum on the right hand side can be computed as is, but that would entail recomputing $\zeta_{q\ge q_1}(N)$ for every possible factorisation of $N = kj$. Change the outer sum in a sum over N and the inner sum over divisors k of N. We may choose $k < N$:

$$\sum_{j=2}^{\infty} \frac{n^j - n}{j} \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln(\zeta_{q\ge q_1}(kj)) \right) = \sum_{N=2}^{\infty} \left(\sum_{k|N,k<N} \mu(k) \left(n^{N/k} - n \right) \right) \frac{\ln(\zeta_{q\ge q_1}(N))}{N} \tag{4.11}$$

One needs suitable values for A, N_{max}, an implementation for $\mu(\cdot), \zeta(\cdot)$ and possibly a sieve to find $k|N$. We may put $\ln(\zeta_{q\ge q_1}(N)) = C \cdot q_1^{-N}$ for some constant C. A rather tight bound on the inner sum only considers its main contribution $k = 1$:

$$\left| \sum_{k|N,k<N} \mu(k)(n^{N/k} - n) \right| < n^N$$

This leads to the following estimate of the summand in the outer sum:

$$\frac{C}{N} \left(\frac{n}{q_1} \right)^N$$

Equating with the required accuracy, say 10^{-m}, gives a relation between q_1 and N_{max}:

$$\ln(N_{max}) + N_{max} \ln\left(\frac{q_1}{n}\right) = m\ln(10) + \ln(C)$$

If one chooses $q_1/n \approx 10$ one would arrive at $N_{max} < m$, if $C = 1$. Note that $C > 1$ and will tend to 1 fast if N increases, so $C = 1$ is a safe choice if the resulting N is large enough. One could set $N_{max} = m$ in order to be on the safe side, or recompute with a larger N_{max} as a check. A simple rough estimate of the remainder, with $N > N_{max}$ may be found by using:

$$C \sum_{N\ge N_{max}+1} \frac{(n/q_1)^N}{N} < \frac{C}{N_{max}+1} \sum_{N\ge N_{max}+1} (n/q_1)^N = \frac{C}{N_{max}+1} \frac{(n/q_1)^{N_{max}+1}}{1-n/q_1} 10^{-m} \approx \frac{n/q_1}{1-n/q_1} 10^{-m}$$

Example - The twin prime constant

We have $n = 2, q_0 = 3$, choose $m = 10$ and assume $C = 1$.

- $q_1 = 5$ Which leads to $N_{max} \approx 21.8$, choose $N_{max} = 22$. Realised significant digits: 10.
- $q_1 = 23$ Which leads to $N_{max} \approx 8.5$, choose $N_{max} = 9$. Realised significant digits: 11.

Example - A_8

We have $n = 16, q_0 = 47$, choose $m = 10$ and assume $C = 1$.

- $q_1 = 53$ Which leads to $N_{max} \approx 16.9$, choose $N_{max} = 17$. Realised significant digits: 10.
- $q_1 = 163$ Which leads to $N_{max} \approx 8.9$, choose $N_{max} = 9$. Realised significant digits: 10.

5 Computed values of $C_{2n}, F_{a_{2n}}$

In order to estimate the number of symmetrical prime constellations A_k less than a given bound x we found that we have (where a different notation is introduced):

$$\pi_{A_n}(x) \doteq \pi_{f_1,\cdots,f_{2n}}(x) = C_f \frac{x}{\ln(x)^{2n}} = C_{A_n} \frac{x}{\ln(x)^{2n}}$$

If n increases, the number of constellations less than x drop. The only hope is that the constant C_f is relatively large to counter this effect. Its value will at least give some insight into the amount of work that is involved to find a constellation or that all hope is lost from the very start.

5.1 $A_2 = [0, 2, 6, 8]$

The number of free residues are:

q	# free residues
2	1
3	1
5	1
7	3

We already have $5 - 4 = 1, 7 - 4 = 3$ free residues, we may take $q_0 = 5$. We thus find:

$$F_8 = \frac{2^3 \cdot 1}{1^4} \frac{3^3 \cdot 1}{2^4} = \frac{27}{2} = 13.5$$

and:

$$C_4 = \prod_{q \in \mathbb{P}, q \geq 5} \frac{q^3(q-4)}{(q-1)^4} \approx .307494878758$$

We find:

$C_{A_2} \approx 4.1511808632$[9]

I chose to include as many terms as possible in the constant C_n. As an alternative one could always choose the maximum value for $q_0 \leq a_{2n}$. That would have been more inline with the choices discussed earlier. The downside is that one should include all these primes in F_{2n}. It is of no consequence to the value of C_{A_n}.

5.2 $A_3 = [0, 2, 12, 14, 24, 26]$

The number of free residues are:

q	# free residues	q	# free residues
2	1	13	8
3	1	17	11
5	1	19	13
7	3	23	17
11	6		

We have $23 - 6 = 17, 19 - 6 = 13, 17 - 6 = 11$ free residues, we may take $q_0 = 17$. We thus find:

$$F_{16} = \frac{2^5 \cdot 1}{1^6} \frac{3^5 \cdot 1}{2^6} \frac{5^5 \cdot 1}{4^6} \frac{7^5 \cdot 3}{6^5} \frac{11^5 \cdot 6}{10^6} \frac{13^5 \cdot 8}{12^6} = 96.2951142008421$$

and:

$$C_6 = \prod_{q \in \mathbb{P}, q \geq 17} \frac{q^5(q-6)}{(q-1)^6} \approx .739097036243^{10}$$

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[9] The on-line encyclopedia of integer sequences, A061642

[10] Compare withThe on-line encyclopedia of integer sequences, A269846

We find:

$C_{A_3} \approx 71.1714335105$

5.3 $A_4 = [0, 2, 12, 14, 18, 20, 30, 32]$

A check reveals that $q_0 = 11$ suffices. We have

$$F_{32} = \frac{2^7 \cdot 1}{1^8} \frac{3^7 \cdot 1}{2^8} \frac{5^7 \cdot 1}{4^8} \frac{7^7 \cdot 2}{6^8} = 1278.306978444258372$$

and:

$$C_8 = \prod_{q \in \mathbb{P}, q \geq 11} \frac{q^7(q-8)}{(q-1)^8} \approx .2324193345867$$

We find:

$C_{A_4} \approx 297.1032573276$

5.4 $A_5 = [0, 2, 6, 8, 18, 20, 30, 32, 36, 38]$

Here $q_0 = 23$ suffices. We have

$$F_{38} = \frac{2^9 \cdot 1}{1^{10}} \frac{3^9 \cdot 1}{2^{10}} \frac{5^9 \cdot 1}{4^{10}} \frac{7^9 \cdot 1}{6^{10}} \frac{11^9 \cdot 2}{10^{10}} \frac{13^9 \cdot 4}{12^{10}} \frac{17^9 \cdot 8}{16^{10}} \frac{19^9 \cdot 10}{18^{10}} \approx 3082.1300015483156102$$

and:

$$C_8 = \prod_{q \in \mathbb{P}, q \geq 23} \frac{q^9(q-10)}{(q-1)^{10}} \approx .561884255151$$

We find:

$C_{A_5} \approx 1731.8003201998$

5.5 A_6 through A_{13}

Instead of giving all the details for the remaining values C_{A_n}, a summary of the results:

n	q_0	F_{2n}	C_n	C_{A_n}
6	17	322292.42	.17608349	56750.37
7	47	625235.66	.63814423	398990.53
8	47	1979786.18	.54635368	1081663.46
9	67	62117967.16	.59989868	37264486.50
$10a$	79	9430329685.26	.60747747	5728712858.35
$10b$	79	1593897580.55	.60747747	968256876.41
11	79	39835337400.00	.54172319	21579725856.15
12	107	567180030239.48	.61136188	346752246821.55
$13a$	113	6900553412737.26	.59628958	4114728064971.66
$13b$	113	11445368568398.75	.59628958	6824753964719.15
$13c$	113	16945940845980.42	.59628958	10104687872946.43
$13d$	113	21105314296815.85	.59628958	12584878902155.73

The constants are accurate up to the implied displayed accuracy. The constellations for A_{13}:

$A_{13a} = [0, 2, 6, 8, 36, 38, 48, 50, 66, 68, 78, 80, 108, 110, 138, 140, 150, 152, 168, 170, 180, 182, 210, 212, 216, 218]$
$A_{13b} = [0, 2, 6, 8, 36, 38, 48, 50, 66, 68, 90, 92, 108, 110, 126, 128, 150, 152, 168, 170, 180, 182, 210, 212, 216, 218]$
$A_{13c} = [0, 2, 6, 8, 36, 38, 66, 68, 78, 80, 90, 92, 108, 110, 126, 128, 138, 140, 150, 152, 180, 182, 210, 212, 216, 218]$
$A_{13d} = [0, 2, 36, 38, 48, 50, 66, 68, 78, 80, 90, 92, 108, 110, 126, 128, 138, 140, 150, 152, 168, 170, 180, 182, 216, 218]$

6 Consecutive primes

A value for p such that the $p + a_i$ are prime, with a_i in A_n does ensure that we found a minimal symmetrical set of twin primes, but it does not ensure that these primes are consecutive.

As shown previously we have that A_4 does not contain the values in the set $B_4 = \{b_i\} = \{8, 24\}$ where $p + b_i$ might be prime. In this case we may choose p from a set of 2 residues modulo 210. For each residu, there is one such number b_i that might be prime as well. We know, by construction, that $p + b_i$ is not divisible by any number in $\{2, 3, 5, 7\}$.

We could extend A_4 by using both values in $\{8, 24\}$ separately and compute the corresponding estimate of the number of constellations as a function of x. We find:

$$\pi_{A_4,8}(x) \approx \pi_{A_4,24}(x) \overset{c}{\sim} 504.1 \frac{x}{\ln(x)^9}$$

We need to subtract these estimates for both elements in $\{8, 24\}$ from π_{A_4}. We therefore estimate that the number of symmetrical set of consecutive twin primes is equal to:

$$\pi_{A_4,\text{consecutive}}(x) \approx \pi_{A_4} - 2\pi_{A_4,8}(x) \overset{c}{\sim} 297.1 \frac{x}{\ln(x)^8} \left(1 - \frac{3.394}{\ln(x)}\right) \tag{6.1}$$

or its logarithmic integral counterpart:

$$\pi_{A_4,\text{consecutive}}(x) \overset{c}{\sim} 297.1 \int_2^x \frac{1}{\ln(t)^8} \mathrm{d}t - 1008.2 \int_2^x \frac{1}{\ln(t)^9} \mathrm{d}t \tag{6.2}$$

The given estimates are quite different for small values of x. The first estimate (6.1) will return positive values for $x \geq 30$ and the second estimate (6.2), is positive for $x \geq 2$. In fact one may estimate that they will approximately differ by 5 percent or less if $x \geq e^{160} \approx 3 \cdot 10^{69}$.

A test reveals that even for $n = 4$ both estimates have difficulty to capture the behaviour of the exact value of $\pi_{A_4,\text{consecutive}}(x)$:

	$\pi_{A_4,\text{consecutive}}(x)$		
x	exact	(6.1)	(6.2)
10^6	1	0.17	1041.13
10^7	1	0.51	1041.83
10^8	5	1.83	1944.19
10^9	13	7.30	1053.22
10^{10}	48	32.06	1091.50
10^{11}	228	151.90	1267.89
10^{12}	1132	767.02	2138.54
$2 \cdot 10^{12}$	1817	1262.53	2827.45

This means that one has to extend this table to rather large x in order to be able to see whether the estimates are any good. For $n > 4$ the density of these prime constellations decreases even further. In fact finding just one prime constellation for $n \geq 8$ could take quite some computing time.

In general this type of computation will get more intricate with increasing n. For A_8 we found that p should fall into one of four possible residue classes $\mod p_6\#$, lets call these p_i. Similar to $n = 4$ there is one number b_i for each of these values of p_i, with $b_i \in B = \{24, 50, 66, 92\}$. Unlike $n = 4$ there is a second set $C = \{6, 12, 20, 26, 90, 96, 110\}$ for which $p_i + c_k$ may not be prime for all p_i and c_k.