On the Puzzle 1134

Alessandro Casini

June 2023

This theorem is not a complete proof of the puzzle, it's just a little step forward.

Theorem. Let n be an integer greater that 2 such that the nth Mersenne number $M_n = 2^n - 1 = (2n+1)q$, where q is a prime integer. Than either n is a Lucasian prime or n and $2n+1$ are both composite.

Proof. We'll prove that, with the theorem's conditions, n is prime iff $2n+1$ is prime.

Assume that $2n+1$ is prime. Then the Mersenne number M_n is a semiprime. In that case it's known that n must be a prime p or a prime square p^2 (for example, see [1]). The latter can't occur, in fact in that case $2n+1 = 2p^2+1$ is divisible by 3 unless $p = 3$, but

$$(2 \cdot 9 + 1) \nmid M_9$$

so $p = 3$ doesn't satisfy the conditions of the theorem and it's ignored. Therefore $2n+1 > 3$ is a multiple of 3, contradicting its primality. It follows than n is equal to a prime number.

Conversely, assume now n prime. We know that $2n+1$ is a divisor of M_n. A famous property of Mersenne number with prime index assert that all prime divisors of M_n are of the form $2kn+1$. Since $2n+1$ is a factor and is the smallest number of that form, then $2n+1$ must be prime.

Moreover, if n and $2n+1$ are odd primes, with $2n+1$ a prime factor of M_n, then $2n+1 \equiv \pm 1 \pmod 8$. Given that n is prime, it follows that the last remainder is -1 and therefore

$$n \equiv 3 \pmod 4.$$

We conclude that if $F(n)$ is prime and one of n and $2n+1$ is prime, then n is a Lucasian prime, i.e. a Sophie Germain prime with $n \equiv 3 \pmod 4$. □

References

[1] A. Cambraia Jr, M. P. Knapp, A. Lemos, B. K. Moriya, and P. H. A. Rodrigues, "On prime factors of mersenne numbers," 2021.