A remark on Puzzle 435

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I send the following result on Puzzle 435,

Proposition. Let m be a solution of

$$\varphi(x)+\sigma(x)=kx \qquad (1)$$

where $k>2$ is an integer.

i) If k is odd then any odd solution m must have the form $4\lambda+1$, for some integer λ.
ii) If m has the form $3\rho+2$ for some integer ρ, then $\varphi(m)\equiv 2k(\bmod 3)$.

Proof. In the papers [1], [2] or [3] we can find the following recursion relation derived by Balth. Van der Pol for the numbers $\sigma(k)$

$$\frac{n^2(n-1)}{12}\sigma(n)=\sum_{k=1}^{n-1}(5k(n-k)-n^2)\sigma(k)\sigma(n-k) \qquad (2)$$

it is easily seen that from (2) I obtain the following congruences

$$n^2(n-1)\sigma(n)\equiv 0(\bmod 3) \qquad (3)$$

$$n^2(n-1)\sigma(n)\equiv 0(\bmod 4) \qquad (4)$$

Thus if I multiply (1) by $m^2(m-1)$ and using the congruences (3) y (4) for an integer $m>1$ I obtain the following congruences

$$\varphi(m)m^2(m-1)\equiv km^3(m-1)(\bmod 3) \qquad (5)$$

$$\varphi(m)m^2(m-1)\equiv km^3(m-1)(\bmod 4) \qquad (6)$$

From the congruence (6) I obtain the proof of i) since if m is odd then $(m^2,4)=1$, thus by Euler's Theorem I obtain

$$\varphi(m)(m-1)\equiv km(m-1)(\bmod 4) \qquad (7)$$

now since m is an odd number greater than 1 both numbers $\varphi(m)$ and $m-1$ are even so $\varphi(m)(m-1)\equiv 0(\bmod 4)$ thus from (7) I obtain

$$km(m-1)\equiv 0(\bmod 4) \qquad (8)$$

Since by hypothesis k is odd then using the simplification rule for congruences I obtain

$$m - 1 \equiv 0 (\bmod 4) \tag{9}$$

that is the first result that I have deduced. For ii) I begin from (5) and using the Fermat's Little Theorem because m has the form $3\rho + 2$ then $(m,3) = 1$ I obtain

$$\varphi(m) \equiv 2k (\bmod 3) \tag{10}$$

and it conclude the proof of the second result.

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Rererences:

[1] Balth van der Pol, *On a Non-linear Partial Differential Equation satisfied by the Logarithm of the Jacobian Theta Functions Theorem with Arithmetical Applications I, II*, Nederl. Akad. Wetensch. Proc., Ser. A, **54** (1951), p. 261-271.
[2] J. Touchard, *On prime numbers and perfect numbers*, Scripta Math. **19** (1953)**,** p. 35-39.
[3] Judy A. Holdener, *A Theorem of Touchard on the Form f Odd Perfect Numbers*, Amer. Math. Monthly **109** (2002), p. 661-663.